

2p-COMMUTATOR ON DIFFERENTIAL OPERATORS OF ORDER p

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ABSTRACT. We show that a space of one variable differential operators of order p admits non-trivial $2p$ -commutator and the number $2p$ here can not be improved.

Let A be an associative algebra over a field K of characteristic 0. Let $f = f(t_1, \dots, t_n)$ be some non-commutative associative polynomial. Say that $f = 0$ is an identity of A if $f(a_1, \dots, a_n) = 0$ for any substitutions $t_i := a_i \in A$. Let s_n be a skew-symmetric associative non-commutative polynomial

$$s_n(t_1, \dots, t_n) = \sum_{\sigma \in \text{Sym}_n} \text{sign } \sigma t_{\sigma(1)} \cdots t_{\sigma(n)}.$$

For example,

$$s_2(t_1, t_2) = t_1 t_2 - t_2 t_1 = [t_1, t_2]$$

is a Lie commutator.

Suppose that an associative commutative algebra U has n commuting derivations $\partial_1, \dots, \partial_k$. A linear span of linear operators of a form $u \partial_{i_1} \dots \partial_{i_p}$, where $1 \leq i_1, \dots, i_p \leq k$, is denoted $D_k^{(p)}(U)$. Let $D_k(U) = \cup_{p \geq 0} D_k^{(p)}(U)$ be space of differential operators on U generated by derivations $\partial_1, \dots, \partial_k$. In case of $k = 1$ we reduce notation ∂_1 to ∂ .

It is known that $D_k(U)$ can be endowed by a structure of associative algebra. A multiplication of the algebra $D(U)$ is given as a composition of differential operators. For example, if $k = 1$, then

$$u \partial^p \cdot v \partial^l = \sum_{s=0}^p \binom{p}{s} u \partial^s(v) \partial^{p+l-s}.$$

Certainly this construction can be easily generalized for algebras with several derivations.

We can consider $D_k^{(p)}(U)$ as a space of differential operators of order p . Well known, that any differential operator of first order is a derivation and a space of derivations $\text{Der}(U) = D_k^{(1)}(U)$ forms Lie algebra under commutator,

$$\begin{aligned} u \partial_i, v \partial_j \in D_k^{(1)}(U) &\Rightarrow s_2(u \partial_i, v \partial_j) = u \partial_i \cdot v \partial_j - v \partial_j \cdot u \partial_i \Rightarrow \\ s_2(u \partial_i, v \partial_j) &= u \partial_i(v) \partial_j - v \partial_j(u) \partial_i \in D_k^{(1)}(U). \end{aligned}$$

Main example of the algebra of differential operators appears in the case $U = K[x_1, \dots, x_k]$ and $\partial_i = \partial/\partial x_i$, $i = 1, \dots, k$, are partial differential operators.

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Recall that action of ∂_i on a monom $x^\alpha = x_1^{\alpha_1} \cdots x_k^{\alpha_k}$, where $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbf{Z}_0^k$, is defined by

$$\partial_i x^\alpha = \alpha_i x^{\alpha - \epsilon_i}.$$

Here \mathbf{Z}_0 is a set of non-negative integers and $\epsilon_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbf{Z}_0^k$ (all components of ϵ except i -th are 0).

Denote by A_k an algebra of differential operators on polynomials algebra $K[x_1, \dots, x_k]$ generated by k commuting derivations $\partial_1, \dots, \partial_k$. The algebra A_k is called *Weyl algebra*. Let $A_k^{(p)} = \langle u \partial^\alpha \mid |\alpha| = p \rangle$ be subspace of A_k consisting differential operators of p -th order.

Let us consider $A_k^{(p)}$ as N -ary algebra under N -ary multiplication s_N ,

$$s_N(X_1, \dots, X_N) = \sum_{\sigma \in \text{Sym}_N} \text{sign } \sigma X_{\sigma(1)} \cdots X_{\sigma(N)}.$$

In general this notion is not correct. Might happen that s_N is not well-defined on $A_k^{(p)}$,

$$s_N(X_1, \dots, X_N) \notin A_k^{(p)}$$

for some $X_1, \dots, X_N \in A_k^{(p)}$. We say that $A_k^{(p)}$ admits N -commutator s_N , if

$$s_N(X_1, \dots, X_N) \in A_k^{(p)}$$

for any $X_1, \dots, X_N \in A_k^{(p)}$.

In [2] it was proved that the space of differential operators of first order $A_n^{(1)}$ in addition to Lie commutator s_2 admits $(n^2 + 2n - 2)$ -commutator and that $s_N = 0$ is identity if $N \geq n^2 + 2n$. Let Mat_n be an algebra of $n \times n$ matrices. Amitsur-Levitzky theorem states that Mat_n satisfies the identity $s_{2n} = 0$ and it is a minimal identity [1]. Note that Weyl algebra has no polynomial identity except associativity. So, to construct non-trivial identities we have to consider smaller subspaces of Weyl algebra.

The aim of our paper is to establish that the space of one variable differential operators of order p admits $2p$ -commutator. The number $2p$ here can not be improved: if $N > 2p$, then $s_N = 0$ is identity on $A_1^{(p)}$; if $N < 2p$, then s_N is not well-defined on $A_1^{(p)}$; if $N = 2p$, then s_N is well-defined on $A_1^{(p)}$ and non-trivial. Obtained $2p$ -ary algebra $A_1^{(p)}$ under multiplication s_{2p} is simple and left-commutative. In particular, the $2p$ -algebra $(A_1^{(p)}, s_{2p})$ is homotopical $2p$ -Lie. To formulate exact result we have to introduce some definitions.

Let us given an n -ary algebra (A, ψ) with n -ary skew-symmetric multiplication $\psi : \wedge^n A \rightarrow A$. Say that A has $(2n-2, 1)$ -type identity (in [3] it is called $(n-1)$ -left commutative) if it satisfies the identity

$$\sum_{\sigma \in S^{(2n-2, 1)}} \text{sign } \sigma \psi(a_{\sigma(1)}, \dots, a_{\sigma(n-1)}, \psi(a_{\sigma(n)}, \dots, a_{\sigma(2n-2)}, a_{\sigma(2n-1)})) = 0$$

Say that (A, ω) satisfies $(1, 2n-2)$ -type identity, if

$$\sum_{\sigma \in S^{(1, 2n-2)}} \text{sign } \sigma \psi(a_1, a_{\sigma(2)}, \dots, a_{\sigma(n-1)}, \psi(a_{\sigma(n)}, \dots, a_{\sigma(2n-1)})) = 0,$$

for any $a_1, \dots, a_{2n-1} \in A$. Here

$$S^{(2n-1, 1)} = \{\sigma \in S_{n-1, n} \mid \sigma(2n-1) = 2n-1\},$$

$$S^{(1,2n-1)} = \{\sigma \in S_{n-1,n} | \sigma(1) = 1\},$$

where

$$S_{n-1,n} = \{\sigma \in S_{2n-1} | \sigma(1) < \dots < \sigma(n-1), \sigma(n) < \dots < \sigma(2n-1)\}$$

is a set of shuffle $(n-1, n)$ -permutations on a set $\{1, 2, \dots, 2n-1\}$. Call n -algebra (A, ψ) *left-commutative* if it satisfies the $(2n-2, 1)$ -type identity. Similarly, it is called *right-commutative* if it has the $(1, 2n-2)$ -type identity. In fact, these two notions are equivalent (Lemma 22).

Say that (A, ψ) is *homotopical n-Lie* [4] if it satisfies the following identity

$$\sum_{\sigma \in S_{n-1,n}} \text{sign } \sigma \psi(a_{\sigma(1)}, \dots, a_{\sigma(n-1)}, \psi(a_{\sigma(n)}, \dots, a_{\sigma(2n-1)})) = 0.$$

For k -ary algebra (A, ψ) with k -multiplication $\psi : \wedge^k A \rightarrow A$ and for a subspace $I \subseteq A$ say that I is *ideal* of A , if $\psi(a_1, \dots, a_{k-1}, b) \in I$, for any $a_1, \dots, a_{k-1} \in A, b \in I$. Say that A is *simple*, if it has no ideal except 0 and A .

In our paper we prove the following result.

Theorem 1. *Let $A_1 = D(K[x])$ be one variable Weyl algebra over a field K of characteristic 0. Then*

- $s_{2p+1} = 0$ is a polynomial identity on $A_1^{(p)}$.
- any polynomial identity of degree no more than $2p$ follows from the associativity one
- s_N is not well-defined on $A_1^{(p)}$ if $N < 2p$
- s_{2p} is well-defined and non-trivial operation on $A_1^{(p)}$
- for any $u_1, \dots, u_{2p} \in K[x]$, the following formula holds

$$s_{2p}(u_1 \partial^p, \dots, u_{2p} \partial^p) = \lambda_p \begin{vmatrix} u_1 & u_2 & \dots & u_{2p} \\ \partial(u_1) & \partial(u_2) & \dots & \partial(u_{2p}) \\ \vdots & \vdots & \dots & \vdots \\ \partial^{2p-1}(u_1) & \partial^{2p-1}(u_2) & \dots & \partial^{2p-1}(u_{2p}) \end{vmatrix} \partial^p,$$

where λ_p is a positive integer

- the $2p$ -algebra $(A_1^{(p)}, s_{2p})$ is simple and left-commutative.

Corollary 2. *If $k > 2p$, then $s_k = 0$ is a polynomial identity on $A_1^{(p)}$.*

Corollary 3. *The $2p$ -algebra $(A_1^{(p)}, s_{2p})$ is right-commutative*

Proof. It follows from Lemma 22.

Corollary 4. *The $2p$ -algebra $(A_1^{(p)}, s_{2p})$ is homotopical $2p$ -Lie.*

Proof. By Corollary 2.2 of [3] the algebra $(A_1^{(p)}, s_{2p})$ is homotopical n -Lie.

Corollary 5. *Any polynomial identity of Weyl algebra A_n follows from the associativity identity.*

This result is known. For example it follows from [5].

Proof. Suppose that A_n has some polynomial identity $g = 0$ that does not follow from associativity identity. We can assume that g is multilinear. Suppose that it has degree $\deg g = d$. Then $g = 0$ induces a polynomial identity for any

subspace of A_n . For example $g = 0$ should be identity for $A_1^{(p)}$. Take p such that $2p > d$. We obtain contradiction with the minimality of identity $s_{2p} = 0$ for $A_1^{(p)}$.

Corollary 6. *Let U be an associative commutative algebra with a derivation ∂ . Then s_{2p} is a $2p$ -commutator of $D^{(p)}(U)$ and $s_N = 0$ is identity on $D^{(p)}(U)$ for any $N > 2p$.*

Proof of theorem 1 is based on super-Lagrangians calculus. We do in next section.

1. SUPER-LAGRANGIANS ALGEBRA

Let \mathbf{Z}_0 be set of non-negative integers, E set of sequences with non-negative integer components, and

$$E_k = \{\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) | 0 \leq \alpha_1 < \alpha_2 < \dots < \alpha_k, \alpha_i \in \mathbf{Z}_0\},$$

$$E_{k,0} = \{\alpha \in E_k | \alpha_1 = 0\},$$

$$E_k(l) = \{\alpha \in E_k | |\alpha| = \sum_{i=1}^k \alpha_i = l\},$$

$$E_{k,0}(l) = \{\alpha \in E_{k,0} | |\alpha| = \sum_{i=1}^k \alpha_i = l\}.$$

We endow E_k by lexicographic order, $\alpha \leq \beta$ if $\alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}$, but $\alpha_i < \beta_i$. This order is prolonged to order on E by $\alpha < \beta$ if $\alpha \in E_k, \beta \in E_l, k < l$.

Let us consider Grassman algebra \mathcal{U} generated by formal symbols $\partial^i(a)$, where $i \in \mathbf{Z}_0$. We suppose that the generator a is odd and the derivation ∂ is even. So, elements $\partial^i(a)$ are odd for any $i \in \mathbf{Z}_0$.

For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in E_k$ set

$$a^\alpha = \partial^{\alpha_1}(a_1) \dots \partial^{\alpha_k}(a_k).$$

The algebra \mathcal{U} is super-commutative and associative,

$$a^\alpha a^\beta = (-1)^{kl} a^\beta a^\alpha.$$

$$a^\alpha (a^\beta a^\gamma) = (a^\alpha a^\beta) a^\gamma,$$

for any $\alpha \in E_k, \beta \in E_l, \gamma \in E_s$. In particular, $a^\alpha a^\beta = 0$, if α and β have common components. For example,

$$a^{(2,3,5)} a^{(1,3)} = 0, \quad a^{(1,2,3,5)} a^{(0,4)} = -a^{(0,1,2,3,4,5)}.$$

Let \mathcal{L} be an algebra of super-differential operators on \mathcal{U} under composition. Then operators of a form $a^\alpha \partial^i$, where $\alpha \in E, i \in \mathbf{Z}_0$, collect a base of \mathcal{L} . Composition of operators is defined as usual

$$u \partial^k \cdot v \partial^l = \sum_{i=0}^k \binom{k}{i} u \partial^i(v) \partial^{k+l-i},$$

where elements $u \partial^i(v) \in \mathcal{U}$ are calculated in terms of super-multiplication in super-algebra \mathcal{U} . For example, if $X = a^{(2,4,5)} \partial^2$ and $Y = a^{(0,1,3)} \partial^3$, then

$$\partial(a^{(0,1,3)}) = a^{(0,2,3)} + a^{(0,1,4)},$$

$$\begin{aligned} \partial^2(a^{(0,1,3)}) &= \partial(\partial(a^{(0,1,3)})) = \partial(a^{(0,2,3)} + a^{(0,1,4)}) = a^{(1,2,3)} + a^{(0,2,4)} + a^{(0,2,4)} + a^{(0,1,5)} = \\ &= a^{(1,2,3)} + 2a^{(0,2,4)} + a^{(0,1,5)}, \end{aligned}$$

and

$$\begin{aligned} X \cdot Y &= a^{(2,4,5)} a^{(0,1,3)} \partial^5 + 2a^{(2,4,5)} \partial(a^{(0,1,3)}) \partial^4 + a^{(2,4,5)} \partial^2(a^{(0,1,3)}) \partial^3 = \\ &= a^{(0,1,2,3,4,5)} \partial^5 + 2a^{(2,4,5)} (a^{(0,2,3)} + a^{(0,1,4)}) \partial^4 + a^{(2,4,5)} (a^{(1,2,3)} + 2a^{(0,2,4)} + a^{(0,1,5)}) \partial^3 = \\ &= a^{(0,1,2,3,4,5)} \partial^5, \end{aligned}$$

since

$$a^{(2,4,5)} a^{(0,2,3)} = a^{(2,4,5)} a^{(0,1,4)} = a^{(2,4,5)} a^{(1,2,3)} = a^{(2,4,5)} a^{(0,2,4)} = a^{(2,4,5)} a^{(0,1,5)} = 0.$$

Let $X = \sum_{i=k}^l X_i \in \mathcal{L}$, where $X_i = (\sum_{\alpha \in E} \lambda_{\alpha,i} a^\alpha) \partial^i$, $k \leq i \leq l$ and $X_k \neq 0$. Take $\beta \in E$ such that $\lambda_{\beta,k} \neq 0$ and $\lambda_{\alpha,k} = 0$ if $\alpha > \beta$. So, X has highest term $\lambda_{\beta,k} x^\beta \partial^k$. Call it *leader* of X and denote $leader(X)$. For example,

$$X = 2a^{(0,1,5)} \partial^2 + 5a^{(1,2,3)} \partial^3 - 3a^{(0,2,4)} \partial^2 \Rightarrow leader(X) = -3a^{(0,2,4)} \partial^2.$$

Denote by U_k a linear span of base elements a^α , where $\alpha \in E_k$. Similarly define linear spaces $U_{k,0}$, $U_k(n)$ and $U_{k,0}(n)$ as linear span of base elements a^α , where correspondingly $\alpha \in E_{k,0}$, $\alpha \in E_k(n)$, and $\alpha \in E_{k,0}(n)$.

Let $U_k^+ \subset U_k$ and $U_k^+(n) \subset U_k(n)$ are subsets generated by linear combinations of e^α with non-negative integer coefficients,

$$\begin{aligned} U_k^+ &= \left\{ \sum_{\alpha \in E_k} \lambda_\alpha a^\alpha \mid \lambda_\alpha \in \mathbf{Z}_0 \right\}, \\ U_k^+(n) &= \left\{ \sum_{\alpha \in E_k(n)} \lambda_\alpha a^\alpha \mid \lambda_\alpha \in \mathbf{Z}_0 \right\}. \end{aligned}$$

Note that $U_k^+, U_k^+(n)$ are semigroups under addition,

$$0 \in U_k^+, 0 \in U_k^+(n),$$

and

$$\begin{aligned} u, v \in U_k^+ &\Rightarrow u + v \in U_k^+, \\ u, v \in U_k^+(n) &\Rightarrow u + v \in U_k^+(n). \end{aligned}$$

Let

$$\begin{aligned} L_k &= \langle a^\alpha \partial^i \mid \alpha \in E_k, i \in \mathbf{Z}_0 \rangle, \\ L_k(n) &= \langle a^\alpha \partial^i \mid i + |\alpha| = n, \alpha \in E_k, i \in \mathbf{Z}_0 \rangle. \end{aligned}$$

Denote by $\mathcal{L}^{(\geq p)}$ a space of differential operators of order no less than p .

Proposition 7. For any $p \geq 0$ the subspace $\mathcal{L}^{(\geq p)}$ generates left-ideal on the algebra \mathcal{L} ,

$$\mathcal{L}\mathcal{L}^{(p)} \subseteq \mathcal{L}^{(p)}.$$

Algebras \mathcal{U} and \mathcal{L} are graded,

$$\begin{aligned} U_k(n)U_l(m) &\subseteq U_{k+l}(n+m), \\ L_k(n)L_l(m) &\subseteq L_{k+l}(n+m), \\ U_k(n)L_l(m) &\subseteq L_{k+l}(n+m), \\ L_k(n)U_l(m) &\subseteq L_{k+l}(n+m), \end{aligned}$$

for any $k, l, n, m \in \mathbf{Z}_0$.

Proof. Evident.

Lemma 8. *Let $p \geq 0$. If $u \in U_k(n)$, then $a\partial^p(u) \in U_{k+1,0}(n+p)$. Moreover, if $u \in U_k^+(n)$, then $a\partial^p(u) \in U_{k+1,0}^+(n+p)$.*

Proof. Our Lemma is an easy consequence of the following statements:

$$u \in U_k(n) \Rightarrow \partial(u) \in U_k(n+1),$$

$$u \in U_k^+(n) \Rightarrow \partial(u) \in U_k^+(n+1).$$

To prove these statements we use induction on p .

For $p = 0$ our statement is trivial. Let $p = 1$. If $u = a^\alpha = \partial^{\alpha_1}(a) \cdots \partial^{\alpha_k}(a)$, then by Leibniz rule $\partial(u)$ is a sum of monoms of a form

$$u_i = \partial^{\alpha_1}(a) \cdots \partial^{\alpha_{i-1}}(a) \partial^{\alpha_i+1}(a) \partial^{\alpha_{i+1}}(a) \cdots \partial^{\alpha_k}(a), \quad 1 \leq i \leq k.$$

If $\alpha_{i+1} = \alpha_i + 1$, then by super-commutativity condition $u_i = 0$. If $\alpha_{i+1} > \alpha_i + 1$, then u_i is a base monom. Therefore, if $\alpha \in E_k(n)$, then $\partial(a^\alpha)$ is a linear combination of base monoms a^β , where $\beta \in E_k(n+1)$ with coefficients that are equal to 0 or 1. Hence

$$u \in U_k(n) \Rightarrow \partial(u) \in U_k(n+1),$$

$$u \in U_k^+(n) \Rightarrow \partial(u) \in U_k^+(n+1).$$

So, base of induction is valid.

Suppose that

$$u \in U_k(n) \Rightarrow \partial^{p-1}(u) \in U_k(n+p-1).$$

Then as we established above

$$\partial^p(u) = \partial(\partial^{p-1}(u)) \in U_k(n+p)$$

By similar reasons

$$u \in U_k^+(n) \Rightarrow \partial^{p-1}(u) \in U_k^+(n+p-1) \Rightarrow \partial^p(u) = \partial(\partial^{p-1}(u)) \in U_k^+(n+p).$$

Lemma 9. *For any $k \in \mathbf{Z}_0$ the k -th power $(a\partial^p)^k \in \mathcal{L}$ is a linear combination with non-negative integer coefficients of operators of a form $a^\alpha \partial^i$, where $\alpha \in E_k$, $|\alpha| + i = pk$ and $i \geq p$.*

Proof. By grading property of \mathcal{U} and \mathcal{L} (Proposition 7) it is clear that $(a\partial^p)^k$ is a linear combination of super-differential operators of a form $a^\alpha \partial^i$, where $\alpha \in E_k(pk - i)$ and $i \geq p$. By Lemma 8 coefficients are non-negative integers.

Lemma 10. *If $N > 2p$, then $(a\partial^p)^N = 0$ and*

$$(a\partial^p)^{2p} = \lambda_p a^{(0,1,2,\dots,2p-1)} \partial^p,$$

for some non-negative integer λ_p .

Proof. If $\alpha \in E_N$, and $N = 2p + 1$, then

$$|\alpha| \geq \sum_{i=0}^{N-1} i = N(N-1)/2 = (2p+1)p = pN.$$

Therefore, by Lemma 9 $(a\partial^p)^{2p+1} = 0$. So, $(a\partial^p)^N = 0$, if $N > 2p$.

If $N = 2p$ and $\alpha \in E_N$ then by the same reasons,

$$|\alpha| \geq p(2p-1),$$

and

$$(a\partial^p)^N = \text{leader}((a\partial^p)^N) = \lambda_p a^{(0,1,\dots,2p-1)} \partial^p,$$

for some $\lambda_p \in \mathbf{Z}_0$.

To prove Theorem 1 we have to establish that $\lambda_p > 0$. It will be done in next section.

2. POSITIVITY OF λ_p

Lemma 11. *Let $\delta(k)$ be maximal element in $E_{k+1,0}(pk)$. Then*

$$\delta(k) = \begin{cases} (0, p-l, p-l+1, \dots, p-1, p+1, \dots, p+l-1, p+l), & \text{if } k = 2l, \\ (0, p-l, p-l+1, \dots, p-1, p, p+1, \dots, p+l-1, p+l), & \text{if } k = 2l+1. \end{cases}$$

Proof. Note first of all that $\delta(k) \in E_{k+1,0}(pk)$. Indeed,

$$|\delta(k)| = \begin{cases} 2pl, & \text{if } k = 2l \text{ is even,} \\ p(2l+1), & \text{if } k = 2l+1 \text{ is odd} \end{cases} \Rightarrow |\delta(k)| = pk.$$

Suppose that $\beta \geq \delta(k)$ for some $\beta = (\beta_1, \dots, \beta_{k+1}) \in E_{k+1,0}(pk)$. Then

$$\beta_2 \geq p-l,$$

where $l = \lfloor n/2 \rfloor$.

For $1 < i \leq k$ let us call $\beta_{i+1} - \beta_i$ as i -rise of β and denote $r_i(\beta)$. If $r_i(\beta) \geq 3$ for some $1 < i \leq k$, then we can find $\gamma \in E_{k+1,0}(pk)$ such that $\beta < \gamma$. Take for example, $\gamma_j = \beta_j$, if $j \neq i, i+1$ and $\gamma_i = \beta_i + 1, \gamma_{i+1} = \beta_{i+1} - 1$. Therefore,

$$r_i(\beta) \leq 2, \quad 1 < i \leq k.$$

If $r_i(\beta) = 2$ for some i then $r_j(\beta) = 1$ for any $j \neq i, 1 < j \leq k$. Let us prove it by contradiction. Suppose that $r_i(\beta) = 2$ and $r_j(\beta) = 2$ for $i \neq j, 1 < i, j \leq k$. Then there exists $\mu \in E_{k+1,0}(pk)$, such that $\beta < \mu$. Take for example, $\mu_s = \beta_s$, if $s \neq i, j$, and $\mu_i = \beta_i + 1, \mu_{j+1} = \beta_{j+1} - 1$.

Let $k = 2l+1$. If $r_{s+1}(\beta) > 1$, for some $0 \leq s \leq k-1$, then $\beta_{2+s} > p-l+s$. Therefore, $|\beta| > \sum_{i=p-l}^{p+l} i > pk$. Hence, $r_i(\beta) = 1$ for any $1 < i \leq k$, and, $\beta = \delta(k)$.

Let $k = 2l$. If $r_{s+1}(\beta) > 1$, for some $0 \leq s \leq l-1$, then $\beta_{2+s} > p-l+s$. Therefore,

$$\begin{aligned} \sum_{i=1}^{s+1} \beta_i &\geq \sum_{i=1}^{s+1} \delta(k)_i, \\ \sum_{j=s+2}^{l+1} \beta_j &> \sum_{j=s+2}^{l+1} \delta(k)_j, \\ \sum_{t=l+2}^{2l+1} \beta_t &\geq \sum_{t=l+2}^{2l+1} \delta(k)_t. \end{aligned}$$

Hence,

$$|\beta| = \sum_{i=1}^{s+1} \beta_i + \sum_{j=s+2}^{l+1} \beta_j + \sum_{t=l+2}^{2l+1} \beta_t > \sum_{i=1}^{s+1} \delta(k)_i + \sum_{j=s+2}^{l+1} \delta(k)_j + \sum_{t=l+2}^{2l+1} \delta(k)_t = |\delta(k)| = pk.$$

If $r_{s+1}(\beta) > 1$, for some $l < s \leq k+1$, then $\beta_{2+s} > p-l+s$, and,

$$\sum_{i=1}^{s+1} \beta_i \geq \sum_{i=1}^{s+1} \delta(k)_i,$$

$$\sum_{j=s+2}^{2l+1} \beta_j > \sum_{j=s+2}^{2l+1} \delta(k)_i.$$

Therefore,

$$|\beta| = \sum_{i=1}^{s+1} \beta_i + \sum_{j=s+2}^{2l+1} \beta_j > \sum_{i=1}^{s+1} \delta(k)_i + \sum_{j=s+2}^{2l+1} \delta(k)_j = |\delta(k)| = pk.$$

Hence

$$r_{s+1}(\beta) > 1 \Rightarrow s = l,$$

and $\beta = \delta(k)$. \square

Recall that $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbf{Z}_0^k$ is called *composition* of n with length k if $\sum_{i=1}^k \alpha_i = n$. Denote by $C_k(n)$ set of compositions of n of length k . For $\alpha \in C_k(n)$ denote by $sort(\alpha)$ the composition α written in non-decreasing order. Note that $sort(\alpha)$ gives us a partition of n . For example, $sort((2, 0, 2, 3, 1)) = (0, 1, 2, 2, 3)$. For $\sigma = (0, \sigma_2, \dots, \sigma_{k+1}) \in E_{k+1,0}(n)$ set $\bar{\sigma} = (\sigma_2, \dots, \sigma_k) \in E_k(n)$.

For $\alpha \in E_k, \beta \in E_{k+1}$ set

$$M(\alpha, \beta) = \{\gamma \in E_k | sort(\alpha + \gamma) = \bar{\beta}\}.$$

For $\alpha \in \mathbf{Z}_0^k, \beta \in \mathbf{Z}_0^l$ define $\alpha \smile \beta \in \mathbf{Z}_0^{k+l}$ as a prepend α to β

$$\alpha \smile \beta = (\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l).$$

Let

$$\mathbf{0}_0 = (),$$

$$\mathbf{0}_i = \underbrace{(0, 0, \dots, 0)}_{i \text{ times}}, \quad i > 0.$$

For $\alpha \in \mathbf{Z}_0^k$ set

$$\binom{|\alpha|}{\alpha} = \prod_{i=1}^k \binom{\alpha_1 + \dots + \alpha_k}{\alpha_1, \dots, \alpha_k} = \frac{(\alpha_1 + \dots + \alpha_k)!}{\alpha_1! \dots \alpha_k!}.$$

Let

$$G_0 = \{()\},$$

$$G_k = \{(i) \smile \mathbf{0}_{i-1} \smile \alpha | \alpha \in G_{k-i}, \quad i = 1, 2, \dots, k\}, \quad k > 0.$$

Example.

$$G_1 = \{(1)\}, \quad G_2 = \{(2, 0), (1, 1)\}, \quad G_3 = \{(3, 0, 0), (2, 0, 1), (1, 2, 0), (1, 1, 1)\}.$$

Lemma 12. *If $k = 2l - 1$ is odd,*

$$M(\delta(k-1), \delta(k)) = \{(p-l+i) \smile \mathbf{0}_{i-1} \smile \alpha \smile \mathbf{0}_{l-1} | \alpha \in G_{l-i}, i = 1, 2, \dots, l\}.$$

If $k = 2l$ is even,

$$M(\delta(k-1), \delta(k)) = \{(p-l) \smile \mathbf{0}_{l-1} \smile \alpha | \alpha \in G_l\}.$$

Proof. Evident.

Example. If $p = 5$, then

$$M(\delta(2), \delta(3)) = M((0, 4, 6), (0, 4, 5, 6)) = \{(4, 1, 0), (5, 0, 0)\},$$

$$M(\delta(3), \delta(4)) = M((0, 4, 5, 6), (0, 3, 4, 6, 7)) = \{(3, 0, 1, 1), (3, 0, 2, 0)\}.$$

Lemma 13.

$$\sum_{\alpha \in G_k} \text{sign}(\alpha + (0, 1, \dots, k-1)) \binom{k}{\alpha} = 1$$

Proof. Induction on k . For $k = 1$ our statement is evident. Suppose that it is true for $k - 1$. Note that

$$G_k = \cup_{i=1}^k \{(i) \sim \mathbf{0}_{i-1} \sim G_{k-i}\}.$$

For $\alpha \in G_{k-i}$,

$$(i) \sim \mathbf{0}_{i-1} \sim \alpha + (0, 1, \dots, k-1) = (i, 1, 2, \dots, i-1, \alpha_1 + i, \dots, \alpha_{k-i} + k-1),$$

and,

$$\text{sign}((i) \sim \mathbf{0}_{i-1} \sim \alpha + (0, 1, \dots, k-1)) = (-1)^{i-1} \text{sign}(\alpha + (0, 1, \dots, k-i-1)).$$

Further, for $\alpha \in G_{k-i}$,

$$\binom{k}{(i) \sim \mathbf{0}_{i-1} \sim \alpha} = \binom{k}{(i) \sim \alpha} = \binom{k}{i} \binom{k-i}{\alpha}.$$

Therefore,

$$\begin{aligned} \sum_{\alpha \in G_k} \text{sign}(\alpha + (0, 1, \dots, k-1)) \binom{k}{\alpha} &= \\ \sum_{i=1}^k \sum_{\alpha \in G_{k-i}} (-1)^{i-1} \text{sign}(\alpha + (0, 1, \dots, k-i-1)) \binom{k}{I} \binom{k-i}{\alpha} &= \\ \sum_{i=1}^k (-1)^{i-1} \binom{k}{i} \sum_{\alpha \in G_{k-i}} \text{sign}(\alpha + (0, 1, \dots, k-i-1)) \binom{k-i}{\alpha} &= \end{aligned}$$

(by inductive suggestion)

$$\sum_{i=1}^k (-1)^{i-1} \binom{k}{i} = 1.$$

Lemma 14.

$$\sum_{i=0}^{l-1} (-1)^i \binom{p}{i} = (-1)^{l-1} \binom{p-1}{l-1}.$$

Proof. Induction on l . If $l = 1$, then our statement is evident. Suppose that it is true for $l - 1 \geq 1$. Then

$$\begin{aligned} \sum_{i=0}^{l-1} (-1)^i \binom{p}{i} &= \sum_{i=0}^{l-2} (-1)^i \binom{p}{i} + (-1)^{l-1} \binom{p}{l-1} = \\ (-1)^{l-1} \binom{p-1}{l-2} + (-1)^{l-1} \binom{p}{l-1} &= (-1)^{l-1} \left(\binom{p}{l-1} - \binom{p-1}{l-2} \right) = \\ &= (-1)^{l-1} \binom{p-1}{l-1}. \end{aligned}$$

Lemma 15. *If $k = 2l - 1$, then*

$$\sum_{i=1}^l \sum_{\alpha \in G_{l-i}} \text{sign}((p-l+i) \smile \mathbf{0}_{i-1} \smile \alpha \smile \mathbf{0}_{l-1} + (0, 1, \dots, l-1, l, \dots, 2l-2)) \binom{p}{(p-l+i) \smile \alpha} = \binom{p-1}{l-1}.$$

If $k = 2l$, then

$$\sum_{\alpha \in G_l} \text{sign}(\alpha + (0, 1, \dots, l-1)) \binom{p}{(p-l) \smile \alpha} = \binom{p}{l}.$$

Proof. Let $k = 2l - 1$. For $\alpha \in G_{l-i}$ let $\Gamma(\alpha) \in \mathbf{Z}_0^{2l-1}$ be defined as

$$\Gamma(\alpha) = (p-l+i) \smile \mathbf{0}_{i-1} \smile \alpha \smile \mathbf{0}_{l-1} + (0, p-l+1, \dots, p-1, p+1, \dots, p+l-1).$$

Note that

$$(1) \quad \Gamma(\alpha) = (p-l+i, p-l+1, \dots, p-l+i-1, \alpha_1+p-l+i, \dots, \alpha_{l-i}+p-1, p+1, \dots, p+l-1).$$

By (1)

$$\text{sort}(\Gamma(\alpha)) = (p-l+1, \dots, p-l+i-1, p-l+i) \smile \text{sort}(\alpha_1+p-l+i, \dots, \alpha_{l-i}+p-1, p+1, \dots, p+l-1).$$

Hence,

$$\text{sort}(\Gamma(\alpha)) = \overline{\delta(k)}, \quad \alpha \in G_{l-i}$$

$$\Updownarrow$$

$$\text{sort}(\alpha_1+p-l+i, \dots, \alpha_{l-i}+p-1, p+1, \dots, p+l-1) = (p-l+i+1, \dots, p-1, p, p+1, \dots, p+l-1).$$

Therefore, the condition $\text{sort}(\Gamma(\alpha)) = \overline{\delta(k)}$ is equivalent to the condition

$$(2) \quad \text{sort}(\alpha_1 + p - l + i, \dots, \alpha_{l-i} + p - 1) = (p - l + i + 1, \dots, p - 1, p).$$

By (1)

$$\text{sign} \Gamma(\alpha) = (-1)^{i-1} \text{sign}(p-l+1, \dots, p-l+i, \alpha_1+p-l+i, \dots, \alpha_{l-i}+p-1, p+1, \dots, p+l-1).$$

Therefore, by (2)

$$(3) \quad \text{sign} \Gamma(\alpha) = (-1)^{i-1} \text{sign}(\alpha_1 + p - l + i, \dots, \alpha_{l-i} + p - 1) = (-1)^{i-1} \text{sign}(\alpha_1, \alpha_2 + 1, \dots, \alpha_{l-i} + l - i - 1).$$

Hence,

$$\sum_{i=1}^l \sum_{\alpha \in G_{l-i}} \text{sign} \Gamma(\alpha) \binom{p}{(p-l+i) \smile \alpha} =$$

(by (3))

$$\sum_{i=1}^l \sum_{\alpha \in G_{l-i}} (-1)^{i-1} \text{sign}(\alpha + (0, 1, \dots, l-i-1)) \binom{p}{l-i} \binom{l-i}{\alpha} =$$

$$\sum_{i=1}^l (-1)^{i-1} \binom{p}{l-i} \sum_{\alpha \in G_{l-i}} \text{sign}(\alpha + (0, 1, \dots, l-i-1)) \binom{l-i}{\alpha} =$$

(by Lemma 13)

$$\sum_{i=1}^l (-1)^{i-1} \binom{p}{l-i} =$$

$$\sum_{j=0}^{l-1} (-1)^{l-j-1} \binom{p}{j} =$$

(by Lemma 14)

$$\binom{p-1}{l-1}.$$

So, our Lemma in case of odd k is proved.

Let $k = 2l$. Then

$$\sum_{\alpha \in G_l} \text{sign}(\alpha + (0, 1, \dots, l-1)) \binom{p}{(p-l) \cup \alpha} =$$

$$\sum_{\alpha \in G_l} \text{sign}((\alpha_1, \alpha_1 + 1, \dots, \alpha_l + l-1)) \binom{p}{l} \binom{l}{\alpha} =$$

$$\binom{p}{l} \sum_{\alpha \in G_l} \text{sign}((\alpha_1, \alpha_1 + 1, \dots, \alpha_l + l-1)) \binom{l}{\alpha} =$$

(by Lemma 12)

$$\binom{p}{l}.$$

Our Lemma is proved completely.

Lemma 16. Let μ_k be the coefficient at $a^{\delta(k-1)}$ of the element $a \partial^p(a^{\delta(k-2)})$, if $k > 1$, and $\mu_1 = 1$. If $1 \leq k \leq 2p$, then

$$\mu_k = \begin{cases} \binom{p}{l}, & \text{if } k = 2l + 1 \text{ is odd,} \\ \binom{p-1}{l-1}, & \text{if } k = 2l \text{ is even.} \end{cases}$$

Proof. Follows from Lemmas 12 and 15.

Example. If $p = 5$, then

k	$\delta(k-1)$	μ_k
1	(0)	1
2	(0, 5)	1
3	(0, 4, 6)	5
4	(0, 4, 5, 6)	4
5	(0, 3, 4, 6, 7)	10
6	(0, 3, 4, 5, 6, 7)	6
7	(0, 2, 3, 4, 6, 7, 8)	10
8	(0, 2, 3, 4, 5, 6, 7, 8)	4
9	(0, 1, 2, 3, 4, 6, 7, 8, 9)	5
10	(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)	1

The following two lemmas can be proved in a similar way as Lemmas 12 and 15.

Lemma 17. Let $\delta_1(k)$ be maximal element in $E_{k+1,0}(pk-1)$. Then

$$\delta_1(k) = \begin{cases} (0, p-l, p-l+2, \dots, p+l-1), & \text{if } k = 2l, \\ (0, p-l, p-l+1, \dots, p, p+2, \dots, p+l), & \text{if } k = 2l+1. \end{cases}$$

Lemma 18. Let γ_k be coefficient at $a^{\delta_1(k-1)}$ of $a\partial^{p-1}(a^{\delta(k-2)})$. Then

$$\gamma_k = p \binom{p-1}{\lfloor (k-2)/2 \rfloor}$$

if $2 \leq k \leq 2p-1$.

Lemma 19. Let ν_k be coefficient at $a^{\delta(k-1)}$ of the element $(a\partial^p)^{k-1}(a)$. Then

$$\text{leader}((a\partial^p)^k) = \nu_k a^{\delta(k-1)} \partial^p.$$

Proof. Follows from Lemma 11.

Lemma 20. For any $0 \leq k \leq 2p$,

$$\nu_k \geq \mu_k \nu_{k-1}.$$

(Definition of μ_k see Lemma 16, and definition of ν_k see Lemma 19).

Proof. By Lemmas 8 coefficient at $a^{\delta(k-1)}$ of the element $(a\partial^p)^{k-1}(a)$ is a non-negative integer that is no less than another non-negative integer $(a\partial^p)^{k-1}(\nu_{k-1}a^{\delta(k-2)})$. By Lemma 16 the last number is equal to $\nu_{k-1}\mu_k$.

Example. Let $p = 3$. Then

$$\mu_1 = 1, \mu_2 = 1, \mu_3 = 3, \mu_4 = 2, \mu_5 = 3, \mu_6 = 1$$

and

$$(a\partial^3)^2 = 3a^{(0,1)}\partial^5 + 3a^{(0,2)}\partial^4 + a^{(0,3)}\partial^3, \\ \text{leader}((a\partial^3)^2) = a^{(0,3)}\partial^3, \quad \nu_2 = 1,$$

$$(a\partial^3)^3 = 18a^{(0,1,2)}\partial^6 + 27a^{(0,1,3)}\partial^5 + 15a^{(0,1,4)}\partial^4 + 3a^{(0,1,5)}\partial^3 + 9a^{(0,2,3)}\partial^4 + 3a^{(0,2,4)}\partial^3, \\ \text{leader}((a\partial^3)^3) = 3a^{(0,2,4)}\partial^3, \quad \nu_3 = 3,$$

$$(a\partial^3)^4 = 126a^{(0,1,2,3)}\partial^6 + 189a^{(0,1,2,4)}\partial^5 + 99a^{(0,1,2,5)}\partial^4 + \\ 18a^{(0,1,2,6)}\partial^3 + 75a^{(0,1,3,4)}\partial^4 + 24a^{(0,1,3,5)}\partial^3 + 6a^{(0,2,3,4)}\partial^3, \\ \text{leader}((a\partial^3)^4) = 6a^{(0,2,3,4)}\partial^3, \quad \nu_4 = 6,$$

$$(a\partial^3)^5 = 432a^{(0,1,2,3,4)}\partial^5 + 432a^{(0,1,2,3,5)}\partial^4 + 108a^{(0,1,2,3,6)}\partial^3 + 90a^{(0,1,2,4,5)}\partial^3, \\ \text{leader}((a\partial^3)^5) = 90a^{(0,1,2,4,5)}\partial^3, \quad \nu_5 = 90,$$

$$(a\partial^3)^6 = 90a^{(0,1,2,3,4,5)}\partial^3. \\ \text{leader}((a\partial^3)^6) = (a\partial^3)^6 = a^{(0,3)}\partial^3, \quad \nu_6 = 90.$$

Lemma 21. For any $X_1, \dots, X_N \in A_1^{(p)}$,

$$s_N(X_1, \dots, X_N) = 0,$$

if $N > 2p$ and

$$s_{2p}(\partial^p, x\partial^p, x^2/2\partial^p, \dots, x^{2p-1}/(2p-1)!\partial^p) = \lambda_p \partial^p.$$

Proof. Suppose that $X_i = u_i \partial^p$, where $u_i \in K[x]$. Let us make specialization of a in super-algebra \mathcal{U} . Take $a = (\sum_{i=1}^N u_i \xi_i) \partial^p$, where ξ_i are odd super-generators. Then

$$(a \partial^p)^N = s_N(u_1 \partial^p, \dots, u_N \partial^p) \xi_1 \cdots \xi_N.$$

By Lemma 10 $(a \partial^p)^N = 0$, if $N > 2p$. Therefore, $s_N = 0$ is identity if $N > 2p$.

Now consider the case $N = 2p$. Set $a = \sum_{i=0}^{2p-1} x^i / i! \xi_{i+1}$ where ξ_i are odd elements and ∂ acts on x^i as usual polynomials, $\partial(x^i) = i x^{i-1}$. Then

$$(a \partial^p)^{2p} = s_{2p}(\partial^p, x \partial^p, x^2 / 2 \partial^p, \dots, x^{2p-1} / (2p-1)! \partial^p) \xi_1 \xi_2 \cdots \xi_{2p}$$

Further,

$$\begin{aligned} a^{(0,1,2,\dots,2p-1)} &= \partial^0(a) \partial^1(a) \cdots \partial^{2p-1}(a) = \\ &= \left(\sum_{i=0}^{2p-1} x^i / i! \xi_{i+1} \right) \left(\sum_{i=0}^{2p-1} x^{i-1} / (i-1)! \xi_{i+1} \right) \cdots (\xi_{2p-1} + x \xi_{2p}) \xi_{2p} = \\ &= \xi_1 \xi_2 \cdots \xi_{2p}. \end{aligned}$$

Therefore, by Lemma 10

$$s_{2p}(\partial^p, x \partial^p, x^2 / 2 \partial^p, \dots, x^{2p-1} / (2p-1)! \partial^p) \xi_1 \xi_2 \cdots \xi_{2p} = (a \partial^p)^{2p} = \lambda_p \xi_1 \xi_2 \cdots \xi_{2p} \partial^p.$$

Hence

$$s_{2p}(\partial^p, x \partial^p, x^2 / 2 \partial^p, \dots, x^{2p-1} / (2p-1)! \partial^p) = \lambda_p \partial^p.$$

3. EQUIVALENCE OF LEFT-COMMUTATIVE AND RIGHT-COMMUTATIVE IDENTITIES

Lemma 22. $(2n-2, 1)$ -type and $(1, 2n-2)$ -type identities are equivalent.

Proof. We have to prove that any n -algebra (A, ψ) with $(2n-2, 1)$ -type identity

$$lcom = 0,$$

where

$$lcom(t_1, \dots, t_{2n-1}) = \sum_{\sigma \in S^{(2n-2, 1)}} sign \sigma \psi(t_{\sigma(1)}, \dots, t_{\sigma(n-1)}, \psi(t_{\sigma(n)}, \dots, t_{\sigma(2n-2)}, t_{\sigma(2n-1)})),$$

satisfies the identity

$$rcom = 0,$$

where

$$rcom(t_1, \dots, t_{2n-1}) = \sum_{\sigma \in S^{(1, 2n-2)}} sign \sigma \psi(t_1, t_{\sigma(2)}, \dots, t_{\sigma(n-1)}, \psi(t_{\sigma(n)}, \dots, t_{\sigma(2n-1)})),$$

and vice versa, any n -ary algebra with identity $rcom = 0$ satisfies also the identity $lcom = 0$.

Let us prove that

$$(4) \quad n rcom(t_1, \dots, t_{2n-1}) = rcom_1(t_1, \dots, t_{2n-1}),$$

$$(5) \quad (n-1) lcom(t_1, \dots, t_{2n-1}) = lcom_1(t_1, \dots, t_{2n-1}),$$

where

$$\begin{aligned} rcom_1(t_1, \dots, t_{2n-1}) &= \\ &= \sum_{i=2}^{2n-1} (-1)^{i+1} lcom(t_1, \dots, \hat{t}_i, \dots, t_{2n-1}, t_i) - (n-1) lcom(t_2, \dots, t_{2n-1}, t_1), \end{aligned}$$

$$lcom_1(t_1, \dots, t_{2n-1}) = \sum_{i=1}^{2n-2} (-1)^{i+1} rcom(t_i, t_1, \dots, \hat{t}_i, \dots, t_{2n-1}) - (n-2) rcom(t_{2n-1}, t_1, \dots, t_{2n-2}).$$

Note that $rcom(t_1, \dots, t_{2n-1})$ and $rcom_1(t_1, \dots, t_{2n-1})$ are skew-symmetric under $2n-2$ variables t_2, \dots, t_{2n-1} . Therefore, it is enough to prove that coefficients at $\psi(t_1, \dots, t_{n-1}, \psi(t_n, \dots, t_{2n-2}, t_{2n-1}))$ and $\psi(t_2, \dots, t_n, \psi(t_1, t_{n+1}, \dots, t_{2n-1}))$ of $rcom(t_1, \dots, t_{2n-1})$ and $rcom_1(t_1, \dots, t_{2n-1})$ are equal.

It is easy to see that, if $n \leq i \leq 2n-1$, then the coefficient at $\psi(t_1, \dots, t_{n-1}, \psi(t_n, \dots, t_{2n-1}))$ of

$$(-1)^{i+1} lcom(t_1, \dots, \hat{t}_i, \dots, t_{2n-1}, t_i)$$

is equal to 1. If $1 \leq i < n$, then this coefficient is 0. Therefore, the coefficient at $\psi(t_1, \dots, t_{n-1}, \psi(t_n, \dots, t_{2n-1}))$ of $rcom_1(t_1, \dots, t_{2n-1})$ is equal to n .

Further, if $n \leq i \leq 2n-1$, then the coefficient at $\psi(t_2, \dots, t_n, \psi(t_1, t_{n+1}, \dots, t_{2n-1}))$ of

$$(-1)^{i+1} lcom(t_1, \dots, \hat{t}_i, \dots, t_{2n-1}, t_i)$$

is equal to 0. If $1 \leq i < n$, then this coefficient is 1. Therefore, the coefficient at $\psi(t_2, \dots, t_n, \psi(t_1, t_{n+1}, \dots, t_{2n-1}))$ of $rcom_1(t_1, \dots, t_{2n-1})$ is equal to 0.

Hence, relation (4) is proved completely.

By similar arguments one establishes (5).

Relations (4) and (5) show that identities $rcom$ and $lcom$ are equivalent.

4. PROOF OF THEOREM 1

By Lemma 21 $s_N = 0$ is identity on $A_1^{(p)}$ if $N > 2p$. By Lemma 20

$$\lambda_p = \nu_{2p} \geq \mu_{2p} \cdots \mu_2 \nu_1 > 0.$$

Therefore, by Lemma 21 $s_{2p} = 0$ is not polynomial identity and s_{2p} induces on $A_1^{(p)}$ a non-trivial $2p$ -commutator.

By Lemma 20 for any $1 \leq k \leq 2p-2$

$$\nu_k \geq \mu_k \cdots \mu_2 \nu_1 > 0.$$

Therefore, by Lemmas 8, 17 and 18 the differential $(p+1)$ -th order parts of $(a\partial^p)^k$ are non-zero for any $2 \leq k \leq 2p-1$. Therefore, s_k is not well-defined on $A_1^{(p)}$.

Suppose that $A_1^{(p)}$ has identity of degree no more than $2p$. Then it has skew-symmetric multi-linear consequence. In particular, it has a skew-symmetric polynomial identity of degree $2p$. But $s_{2p} = 0$, as we mentioned above, is not identity. Contradiction.

Suppose that I is a non-trivial ideal of $A_1^{(p)}$ under $2p$ -commutator s_{2p} . Take $0 \neq X = u\partial^p \in I$ with minimal degree $s = \deg u$. Let us prove that $s = 0$ and $X = \eta\partial^p \in I$ for some $0 \neq \eta \in K$. Suppose that it is not true, and $s > 0$. If $s \geq 2p-1$, then by Lemma 21

$$s_{2p}(\partial^p, x\partial^p, \dots, x^{2p-2}\partial^p, X) = \lambda_p \binom{s}{2p-1} \prod_{i=0}^{2p-1} i! x^{s-2p+1} \partial^p \in I,$$

or,

$$x^{s-2p+1} \partial^p \in I.$$

We obtain contradiction with minimality of s . If $0 < s < 2p - 1$, then

$$s_{2p}(\partial^p, x\partial^p, \dots, x^{s-1}\partial^p, X, x^{s+1}\partial^p, \dots, x^{2p-1}\partial^p) = \lambda_p \prod_{i=0}^{2p-1} i! \partial^p \in I,$$

or,

$$\partial^p \in I.$$

Once again we obtain contradiction with minimality of s .

So, we establish that $X = \eta\partial^p \in I$, for some $0 \neq \eta \in K$. Then for any $l \geq 0$,

$$s_{2p}(X, x\partial, \dots, x^{2p-2}\partial^p, x^{l+2p-1}\partial^p) = \eta\lambda_p \binom{l+2p-1}{2p-1} \prod_{i=0}^{2p-1} i! x^l \partial^p \in I.$$

In other words, $x^l \partial^p \in I$ for any $l \geq 0$. This means that $I = A_1^{(p)}$. So, $(A_1^{(p)}, s_{2p})$ is simple $2p$ -algebra.

By Theorem 1.1 (ii) of [3] the algebra $(A_n(p), s_{2p})$ is left-commutative. Presentation of $2p$ -commutator as a Vronskian up to scalar λ_p follows from Lemma 21.

5. EXPRESSIONS FOR λ_p

In this section we give some formulas for λ_p . For $s > 0$ let us define a polynomial

$$f_s(x_1, \dots, x_{2p-1}) = \frac{\sum_{\sigma \in \text{Sym}_{2p}} \text{sign } \sigma (x_{\sigma(1)}(x_{\sigma(1)} + x_{\sigma(2)}) \cdots (x_{\sigma(1)} + x_{\sigma(2)} + \cdots + x_{\sigma(2p-1)}))}{\prod_{1 \leq i < j \leq 2p} (x_i - x_j)}.$$

Then $f_s(x_1, \dots, x_{2p-1})$ is a symmetric polynomial of degree $(2p-1)(s-p)$. In particular, $f_p(x_1, \dots, x_{2p-1}) = \lambda_p$ is constant. The number λ_p appears in calculating of $2p$ -commutator,

$$s_{2p}(u_1 \partial^p, \dots, u_{2p} \partial^p) = \lambda_p \begin{vmatrix} u_1 & u_2 & \cdots & u_{2p} \\ \partial(u_1) & \partial(u_2) & \cdots & \partial(u_{2p}) \\ \vdots & \vdots & \cdots & \vdots \\ \partial^{2p-1}(u_1) & \partial^{2p-1}(u_2) & \cdots & \partial^{2p-1}(u_{2p}) \end{vmatrix} \partial^p.$$

Then

$$\lambda_p = \frac{\sum_{\sigma \in \text{Sym}_{2p}} \text{sign } \sigma (\sigma(1)(\sigma(1) + \sigma(2)) \cdots (\sigma(1) + \sigma(2) + \cdots + \sigma(2p-1)))^p}{\prod_{1 \leq i < j \leq 2p} (i - j)}.$$

For example,

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 90, \lambda_4 = 586656, \lambda_5 = 1915103977500.$$

$$\lambda_6 = 7886133184567796056800.$$

Another way to calculate λ_p . Let \mathcal{M}_p be set of matrices $M = (m_{i,j})$ of order $(2p-1) \times (2p-1)$ such that

- $m_{i,j} \in \mathbf{Z}_0$
- $m_{i,j} = 0$ if $i > j$
- sums by rows are constant, $\sum_{j=1}^{2p-1} m_{i,j} = p$ for any i

- sums by columns $r_j = \sum_{i=1}^{2p-1} m_{i,j}$, are positive and different for all $j = 1, 2, \dots, 2p-1$.

In particular,

$$M = (m_{i,j}) \in \mathcal{M}_p \Rightarrow m_{1,1} = r_1 > 0 \text{ and } m_{2p-1,2p-1} = p.$$

For $M \in \mathcal{M}_p$ denote by $r(M)$ the permutation $r_1 \dots r_{2p-1}$ constructed by column sums.

Example. $p = 2$. Then

$$\mathcal{M}_2 = \left\{ A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \right\}.$$

$$r(A) = 123, r(B) = 123, r(C) = 132, r(D) = 213. \quad \square$$

If $M \in \mathcal{M}_p$, then a sequence $r_1 \dots r_{2p-1}$ induces a permutation, where $r_i = \sum_j m_{i,j}$ are sums by columns. In particular, $1 \leq r_i \leq 2p-1$ for any $1 \leq i \leq 2p-1$. Then

$$\lambda_p = \sum_{M \in \mathcal{M}_p} \text{sign } r(M) \prod_{i=1}^{2p-1} \binom{p}{m_{i,1}, \dots, m_{i,2p-1}},$$

$$\lambda_p = \frac{p!^{2p-1}}{\prod_{j=1}^{2p-1} j!} \sum_{M \in \mathcal{M}_p} \text{sign } r(M) \prod_j \binom{r_j}{m_{1,j}, \dots, m_{j,j}}.$$

Here

$$\binom{n}{n_1, \dots, n_k} = \frac{n!}{n_1! \dots n_k!}$$

is a multinomial coefficient.

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